Proof of Lemma 21.27. The lemma is true when $m = 3$ and $m = 4$. We proceed by induction on m. If G is separable, then each block B of G has an even subgraph whose 2-closure is $E(B)$, by induction. The union of these subgraphs is an even subgraph C whose 2-closure is E. We may therefore assume that G is nonseparable.

If G is not simple, let e and f be parallel edges. We may assume that e and f lie in a 3-edge cut. Otherwise, $G \setminus e$ is 3-edge-connected and, by induction, has an even subgraph C whose 2-closure is $E \setminus \{e\}$. Then C is an even subgraph of G whose 2-closure is E. Because $m \geq 5$, the graph $G' := G / \{e, f\}$ is 3-edge-connected. By induction, G' has an even subgraph C' whose 2-closure is $E \setminus \{e, f\}$. Then either $C' \cup \{e\}$ or $C' \cup \{e, f\}$ is an even subgraph whose 2-closure is E. We may therefore assume that G is simple.

Let C be an even subgraph of G such that:

i) the subgraph H of G induced by the 2-closure of C is connected,

ii) subject to (i), C is as large as possible.

We shall show that H is a spanning subgraph of G and hence, by the definition of 2-closure, that $H = G$.

If not, let K be a component of $G - V(H)$. Because G is 3-edge-connected, there are at least three edges linking H and K. We shall show that two of those edges are incident with vertices in some 2-edge-connected subgraph F of K.

If K itself is 2-edge-connected, then we take F to be K. Otherwise, for some cut edge e of K, at least one of the two components of $K \setminus e$ is 2-edge-connected, and we take F to be such a subgraph of K. In either case, $|\partial_K(F)| \leq 1$. By the 3-edge-connectivity of G, there are (at least) two edges linking H and F.

The ends in F of these two edges are distinct because no vertex of K is adjacent to two vertices of H, by the definition of 2-closure. Moreover, these ends are connected by two edge-disjoint paths in F, by the edge version of Menger's Theorem (7.17). Let S be the union of the edge sets of these paths. Then $C \cup S$ contradicts the choice of C. This contradiction establishes the lemma. ■