Appendix B

Hints to Selected Exercises

Section 1.1

1.1.2 (b) Observe that a simple bipartite graph with the maximum number of edges is necessarily complete bipartite and that $K_{r,s}$ has rs edges. Now show that if $s - r \geq 2$, then $K_{r+1,s-1}$ has more edges than $K_{r,s}$. Alternatively, show that $n^2/4 - r(n-r)$ is a perfect square, and thus nonnegative.

1.1.7 (b) To determine $e(Q_n)$ apply Theorem 1.1.

1.1.9 (a) Counting in two ways (see inset).

1.1.10 Find upper bounds on the degrees and apply Theorem 1.1.

1.1.11 This is a generalization of Exercise 1.1.2. Use a similar proof technique.

1.1.12 If G is *not* connected, there is a partition (X, Y) of V such that no edge of G joins a vertex in X to a vertex in Y. What is the largest number of edges that G can have if $|X| = r$ and $|Y| = n - r$?

1.1.13 Observe that if a graph G is disconnected, then there exists a subset X of V, with $|X| \leq \lfloor n/2 \rfloor$, such that no edge of G joins a vertex in X to a vertex in $V \setminus X$.

1.1.15 Observe that a set of vectors over $GF(2)$ is linearly dependent if and only if it contains a subset whose sum is the zero vector.

1.1.16

(a) Necessity: apply Theorem 1.1.
Sufficiency: how many d_i's are odd?
(b) Necessity: apply Theorem 1.1.
Sufficiency: induction on $(\sum_{i=2}^{n} d_i) - d_1$.

1.1.18 (b) Counting in two ways. Let $G = (V, E)$, where $V = \{v_1, v_2, \ldots, v_n\}$ and $d(v_i) = d_i$, $1 \leq i \leq n$. Set $X := \{v_1, v_2, \ldots, v_k\}$. For $v_i \in X$, find a lower bound on the number of edges joining v_i to vertices in $V \setminus X$. For $v_i \in V \setminus X$, find an upper bound on the number of edges joining v_i to vertices in X. Deduce lower and upper bounds on the number of edges with one end in X and the other in $V \setminus X$.

1.1.19 (a) To prove the necessity, first show that if G is a simple graph with $u_1v_1, u_2v_2 \in E$ and u_1v_2, $u_2v_1 \notin E$, then $(G \setminus \{u_1v_1, u_2v_2\}) + \{u_1v_2, u_2v_1\}$ has the same degree sequence as G. Deduce that if $\mathbf{d}$ is graphic, then there is a simple graph G with vertex set $\{v_1, v_2, \ldots, v_n\}$ such that $d(v_i) = d_i$, $1 \leq i \leq n$, and $N(v_1) = \{v_2, v_3, \ldots, v_{d_1+1}\}$.

1.1.20 Define a graph on S in which x_i and x_j are adjacent if and only if they are at distance one. How large can the degree of a vertex be in this graph?

1.1.21

(a) This is a classical result of linear algebra, and is true for any real symmetric matrix.
(b) This is true for any integer square matrix $\mathbf{A}$. Suppose that $\lambda := p/q$ is an eigenvalue of $\mathbf{A}$, where p and q are coprime integers. Assume that $q > 1$. Using the fact that a scalar multiple of an eigenvector is also an eigenvector, show that there exists an integral eigenvector corresponding to λ not all of whose coefficients are divisible by q. Derive a contradiction.

1.1.24 (a) Consider the case where $G = (V, E)$ is simple. Let $V = \{v_1, v_2, \ldots, v_n\}$. An eigenvector of G corresponding to an eigenvalue λ may be viewed as a weight function on V such that the sum of the weights on the neighbours of a vertex is equal to λ times the weight of the vertex. Choose an eigenvector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ of λ such that $|x_j| := \max\{|x_i| : 1 \leq i \leq n\} = 1$, and consider the vertex v_j.

Section 1.2

1.2.14 (a) Use Exercise 1.2.11 to reduce the number of cases to consider.

1.2.16

(c) How many edges has a self-complementary graph?
(d) An isomorphism mapping G to $\overline{G}$ induces a pairing of the vertices of G. What can you say about their degrees?

1.2.17 Since G is not vertex-transitive, there exist vertices x and y such that no automorphism of G maps x to y. Let X and Y denote the orbits of x and y, respectively. Consider a neighbour z of x. Using the fact that G is edge-transitive and that y is not isolated, show that $z \in Y \setminus X$. Now show that (X, Y) is a bipartition of G.

1.2.18 (b) To show that the Folkman graph is not vertex-transitive, observe that it is bipartite. Find a property of the vertices in one part that is not shared by the vertices in the other part.

1.2.20 By Exercise 1.2.8, an automorphism of G can be regarded as a permutation matrix $\mathbf{P}$ such that $\mathbf{PAP}^t = \mathbf{A}$. Using the fact that $\mathbf{P}^t = \mathbf{P}^{-1}$ (why?), show that if $\mathbf{x}$ is an eigenvector of G corresponding to an eigenvalue λ, then so is $\mathbf{Px}$. Now use the fact that G has n distinct eigenvalues, that eigenvectors corresponding to distinct eigenvalues are orthogonal (why?), and that $\mathbf{P}$ is a permutation matrix, to deduce that $\mathbf{Px} = \pm\mathbf{x}$. Conclude.

Section 1.3

1.3.7 (b) For each of the four triangles in the graph of Figure 1.20, apply (a) to the set of three intervals represented by its vertices. Deduce that one of the six intervals must meet all five of the other intervals.

1.3.15 (a) Assuming that $|X| \geq |Y|$, show that

$$\frac{1}{|X|(|Y| - d(x))} \geq \frac{1}{|Y|(|X| - d(y))}$$

for all $xy \notin E$. Now apply the proof technique of counting in two ways to deduce that $|X| = |Y|$.

1.3.17 (a)
(i) The complement of $L(K_4)$ is a 1-regular graph on six vertices. Use this fact to show that $\text{Aut}(L(K_4))$ and $\text{Aut}(K_4)$ have different orders.
(ii) Suppose that $n \geq 3$, $n \neq 4$. Show that an automorphism of K_n induces an automorphism of $L(K_n)$. Deduce that the symmetric group S_n is a subgroup of $\text{Aut}(L(K_n))$. Noting that an automorphism of $L(K_n)$ maps adjacent edges of K_n to adjacent edges, show that every automorphism of $L(K_n)$ is induced by an automorphism of K_n.

1.3.18 (c) Up to isomorphism, there are just two groups of order ten, namely $\mathbb{Z}_{10}$ and $\mathbb{Z}_2 \times \mathbb{Z}_5$. Proceed by considering various cases (see Holton, D.A. and Sheehan, J. (1993). *The Petersen Graph.* Cambridge University Press, pp. 292–293).

Section 1.5

1.5.7 (a) Let $\mathbf{B}$ be an $r \times r$ submatrix of $\mathbf{M}$. If $r = 1$, $\det \mathbf{B} = 0, \pm 1$. Suppose that $r \geq 2$. If $\mathbf{B}$ has a column of 0s, or if each column of $\mathbf{B}$ has two nonzero entries, then $\det \mathbf{B} = 0$ (why?). If not, proceed by induction.

1.5.11 (b) Let xy be a fixed edge of G. First, show that there is no automorphism of G which maps the ordered pair (x, y) to its converse (y, x). Then show that, for every edge uv of G, the orbit of the ordered pair (x, y) under the automorphism group of G includes exactly one of the two ordered pairs (u, v) and (v, u) but not both. Now obtain a directed graph $\vec{G}$ by orienting an edge uv from u to v if (u, v) is in the orbit of (x, y) and from v to u, otherwise. Show that every automorphism of G is also an automorphism of $\vec{G}$. Conclude that $\vec{G}$ is vertex-transitive and hence diregular. Deduce that G is regular of even degree.

1.5.12 Show that $\mathbf{B}$ is skew-symmetric, and hence that $\det \mathbf{B} = 0$ when n is odd. When n is even, use the fact that the number $d(n)$ of derangements (permutations in S_n with no fixed point) is odd (in fact, $d(n) = nd(n-1) + (-1)^n$ for $n \geq 2$) to deduce that $\det \mathbf{B} \neq 0$.

Section 2.1

2.1.2 (a) Consider the ends of a nontrivial maximal path.

2.1.3 (a) If G has a vertex of degree less than two, delete it and apply induction.

2.1.4 (a) Consider a maximal path and the neighbours of one end of the path.

2.1.5 (a) Consider a maximal path and the neighbours of one end of the path.
(b) Apply induction.

2.1.6 Consider a maximal path and the neighbours of one end of the path.

2.1.8 (a) Consider the neighbours of two adjacent vertices.

2.1.9 Consider a vertex, its neighbours, and the neighbours of its neighbours.

2.1.11 (a) Consider the ends of a maximal directed path.

2.1.12 Consider the arcs in a longest directed path.

2.1.14 (a) Show that G is regular. Now consider the degrees in a vertex-deleted subgraph.

2.1.17 (a) Consider a shortest odd cycle.

2.1.18 Let X be a set of p vertices which induce a subgraph $G[X]$ whose minimum degree is as large as possible. If there is a vertex u of $G[X]$ whose degree is less than q, and also a vertex v of $G - X$ whose degree is less than $(k-1)q$, consider the set $(X \setminus \{u\}) \cup \{v\}$, and apply induction on k.

2.1.19 Let C be a shortest odd cycle in $KG_{m,n}$. Show that the union of the $(n-2m)$-sets $S \setminus \{x \cup y\}$, $xy \in E(C)$, is the entire n-set S.

2.1.22 Consider the set of minimal elements of P. Show that they form an antichain, delete them, and apply induction on $|X|$.

2.1.23 (a) Show that any vertex of degree three or more in G has a neighbour of degree one, and apply induction on n.

Section 2.2

2.2.11 Consider a shortest path between two nonadjacent vertices.

2.2.18 (a) By contradiction. Let G be a smallest counterexample. Show that the girth of G is at least five, and that $\delta \geq 3$. Deduce that $n \leq 8$. Conclude that no such graph exists.

2.2.19 One may assume that G is connected (why?). Consider the cases $\delta \geq 3$ and $\delta \leq 2$, and apply induction.

2.2.20 (a) Show that either G has a vertex of odd degree such that $G - x$ is connected, or G has an edge xy such that $G - \{x, y\}$ is connected, and apply induction.

2.2.21 (b) Does the 'Proof' take into account all the graphs satisfying the hypotheses of the 'Theorem'?

2.2.25 Form a simple graph with vertex set S, two points being adjacent if and only if their distance is exactly one. Show that any vertex of degree three or more in this graph has a neighbour of degree one, and apply induction.

2.2.27 Suppose that all induced subgraphs of G on k vertices have l edges. Show that, for any two vertices v_i and v_j,

$$\begin{aligned} e(G) - d(v_i) &= e(G - v_i) = l\binom{n-1}{k}/\binom{n-3}{k-2} \\ e(G) - d(v_i) - d(v_j) + a_{ij} &= e(G - v_i - v_j) = l\binom{n-2}{k}/\binom{n-4}{k-2} \end{aligned}$$

where $a_{ij} = 1$ or 0 according as v_i and v_j are adjacent or not. Deduce that a_{ij} is independent of i and j.

Section 2.4

2.4.6 (a) Label the vertices $0, 1, 2, ..., 2n$. Arrange the vertices $1, 2, ..., 2n$ in a circle with 0 at the centre. Consider the cycle $(0, 1, 2, 2n, 3, 2n-1, 4, 2n-2, ..., n+2, n+1, 0)$ and its rotations.

2.4.7 Denote by G the graph in Figure 2.7, and by P_1 and P_2 its inner and outer pentagons. Consider a hypothetical cycle decomposition $\mathcal{C}$ of G, no member of which is a 2-cycle. Show that $|E(C) \cap E(P_i)| = 2$, $i = 1, 2$. Derive a contradiction based on parity.

2.4.9 Consider an arbitrary linear ordering of the vertices of the digraph.

2.4.10 (a) Set $P := \{v_1, \dots, v_n\}$ and $\mathcal{L} := \{L_1, \dots, L_m\}$, Denote by $\mathbf{m}_j$ the jth column of $\mathbf{M}$, and consider the vectors $\mathbf{x}_i := \sum\{\mathbf{m}_j : v_i \in L_j\}$ and $\mathbf{y}_i := \sum\{\mathbf{m}_j : v_i \notin L_j\}$, $1 \le i \le n$. Show that $\mathbf{x}_i$ has ith coordinate $d_i := d(v_i)$ and all other coordinates 1, and that $\mathbf{y}_i$ has ith coordinate 0 and kth coordinate $d_k - 1$, $k \neq i$. Express each unit vector of $\mathbb{R}^n$ as a linear combination of the vectors $\mathbf{x}_i$ and $\mathbf{y}_i$, $1 \le i \le n$.

Section 2.5

2.5.1 (a) Counting in two ways. Consider the submatrix of the incidence matrix $\mathbf{M}$ consisting of the rows corresponding to the vertices in X.

Section 2.6

2.6.6 If G is even, set $X := V$. If not, proceed by induction on n. Let v be a vertex of odd degree in G. Consider the graph obtained from $G - v$ by replacing the subgraph $G[N(v)]$ by its complement.

Section 2.7

2.7.1 Use Kelly's Lemma (2.20) to obtain constraints on the numbers of subgraphs of various types in the missing vertex-deleted subgraph.

2.7.3 (b) A tree is a connected acyclic graph. Look for a tree on eleven vertices containing a path of length eight.

Section 3.1

3.1.1 Consider a shortest xy-walk.

3.1.2 Use induction on k.

3.1.8 Let x be a vertex of degree $n-2$, y the vertex nonadjacent to x, and X and Y the neighbour sets of x and y, respectively. Consider the set $X \setminus Y$.

Section 3.2

3.2.3 (a) Review the theorems in Section 2.5.

3.2.4 Proceed by contradiction. Assume that there is a cut edge, delete it, and consider the degrees of vertices in the resulting subgraph.

Section 3.3

3.3.5 Add a new vertex with neighbour set X.

3.3.7 If $G-v$ contains a cycle C, consider a suitable Euler tour of the component of $G \setminus E(C)$ containing v. If $G-v$ is acyclic, let Q be a v-trail of G which is not an Euler tour. Show that $G \setminus E(Q)$ has exactly one nontrivial component.

3.3.9 Let $W := v_0e_1v_1e_2v_2 \ldots e_mv_m$ be an Euler tour of a graph G, where $v_m = v_0$. Suppose that $v_i = v_0$, where $0 < i < m$. Show that $v_0Wv_ie_mv_{m-1}\overleftarrow{W}v_i$ is also an Euler tour of G.

Section 3.4

3.4.7 Apply Rédei's Theorem (2.3) to an appropriate digraph defined on the same vertex set.

3.4.11 (b) Suppose that D is strong and contains an odd cycle $v_1v_2 \ldots v_{2k+1}v_1$. If $(v_i, v_{i+1}) \in A$, set $P_i := (v_i, v_{i+1})$; if $(v_i, v_{i+1}) \notin A$, let P_i be a directed (v_i, v_{i+1})-path. If all the paths P_i are of odd length, apply (a).

Section 3.5

3.5.3 Use Exercise 2.2.6.

3.5.4 (b) Consider all possible symmetric differences of the cycles C_1, C_2, and C_3.

3.5.7 Induction on m. If G has an edge cut of size two, contract one of these edges. If not, apply the induction hypothesis to each edge-deleted subgraph of G, and combine appropriate numbers of copies of the m uniform cycle covers thereby obtained.

3.5.8 (a)(i) To prove sufficiency, reduce to the case in which G has no edge cut of size one or two, and apply Exercise 3.5.7 to each edge-deleted subgraph of G. Obtain a positive integer r such that each of the vectors $\mathbf{u}_e$, $e \in E$, defined by

$$\mathbf{u}_e(f) := \begin{cases} 0 \text{ if } f = e \\ r \text{ if } f \in E \setminus \{e\} \end{cases}$$

is a linear combination of vectors in $\mathbf{F}_{\mathcal{C}}$.

Section 4.1

4.1.4 Consider each component of F separately.

4.1.7 To prove sufficiency, consider a graph G with degree sequence $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ and as few components as possible. If G is not connected, show by a suitable exchange of edges (as in the hint to Exercise 1.1.19) that there is a graph with degree sequence $\mathbf{d}$ and fewer components than G.

4.1.12 Show that G has a spanning forest with the same set of labels as G. Apply Exercise 4.1.4 to deduce that some element of S is not a label of G. Conclude.

4.1.14 (b) Form a bipartite graph $B(X, Y)$, where $X := \{x_1, x_2, \ldots, x_m\}$ is the set of pizzas, $Y := \{y_1, y_2, \ldots, y_n\}$ is the set of students, and x_i is joined to y_j if a portion of pizza x_i is allotted to student y_j. Show that B has no more than d components.

4.1.15 Delete the root and use induction on n.

4.1.19 (a) Let T be a tree. Show by induction that there is a Hamilton cycle in T^3 that uses any prescribed edge $e = xy$ of T. Apply the induction hypothesis to the two components of $T \setminus e$, choosing an appropriate edge in each component.

Section 4.2

4.2.5

a) Denote by F'_n the graph obtained from $F_n := P_{n-1} \vee K_1$ by doubling the edge from the vertex of K_1 to one end of P_{n-1}. Set $f_n := t(F_n)$ and $f'_n := t(F'_n)$. Apply Proposition 4.9 to derive two recurrence relations involving these two functions. Deduce that $f_n - 3f_{n-1} + f_{n-2} = 0$, and solve this recurrence relation.

b) Denote by W'_n the graph obtained from W_n by deleting a spoke. Set $w_n := t(W_n)$ and $w'_n := t(W'_n)$. Observe that each spoke of W_n is in the same number a_n of spanning trees. Likewise, each rim edge of W_n is in the same number b_n of spanning trees. Find three relations linking a_n, b_n, w_n, w'_n, and f_n, and apply Proposition 4.9 to an appropriate edge e of W'_n to obtain another relation linking these quantities. Deduce from the resulting four relations that $w_n - w_{n-1} = f_n + f_{n-1}$, and apply (a) to obtain a recurrence relation for w_n. Solve this recurrence relation.

4.2.11 To each labelled tree T with vertex set $\{1, 2, \ldots, n\}$, associate the sequence $(j_1, j_2, \ldots, j_{n-2})$, where j_1 is the neighbour in $T_1 := T$ of the smallest leaf i_1 of T_1, j_2 is the neighbour in $T_2 := T_1 - i_1$ of the smallest leaf i_2 of T_2, and so on.

4.2.12

a) The multinomial coefficients satisfy the following recursion:

$$\binom{n}{d_1, d_2, \ldots, d_k} = \sum_{i=1}^{k} \binom{n-1}{d_1, \ldots, d_i - 1, \ldots, d_k}$$

Using the fact that every tree on two or more vertices has a vertex of degree one, show that $t(n; d_1, d_2, \ldots, d_n)$ satisfies a similar recursion, and apply induction on n.

b) The Multinomial Theorem states that

$$(x_1 + x_2 + \cdots + x_k)^n = \sum \binom{n}{d_1, d_2, \ldots, d_k} x_1^{d_1} x_2^{d_2} \ldots x_k^{d_k}$$

where the sum is taken over all partitions of n into k nonnegative integers $d_1, d_2, \ldots, d_k$. Consider the expansion of $(\sum_{i=1}^{n} 1)^{n-2}$.

4.2.13 Call the vertices in the m-set of $K_{m,n}$ *red* and the remaining n vertices *blue*. Observe that each spanning tree of $K_{m,n}$ rooted at a red vertex is the union of a spanning forest F of stars rooted at red vertices and a set S of $m-1$ blue–red edges. How many choices are there for F and for S?

Section 4.3

4.3.9 Let C be a Hamilton cycle of G and e an edge of C. Apply Corollary 4.12 to $T := C \setminus e$.

4.3.10 Apply Corollary 4.12.

Section 5.1

5.1.1 Consider the ends of a nontrivial maximal path.

5.1.4 Add a new vertex x adjacent to each vertex of X, and a new vertex y adjacent to each vertex of Y. Apply Exercise 5.1.1 and Theorem 5.1.

5.1.5 Let C_1 and C_2 be two longest cycles. Apply Exercise 5.1.4 with $X := V(C_1)$ and $Y := V(C_2)$.

Section 5.2

5.2.3 (a). To show that $\overset{C}{\sim}$ is transitive, consider three edges e, f, and g of G. Let C_1 be a cycle containing e and f, and C_2 a cycle containing f and g. Show that $C_1 \cup C_2$ is nonseparable and apply Theorem 5.2.

5.2.5 Show that the edges of a cycle lie in a single block of the graph. Now use Theorem 2.7 for (a) and Theorem 4.7 for (b).

5.2.7 (a) If G is not bipartite it contains an odd cycle C. Apply Exercise 5.1.4 with $X := \{x, y\}$ and $Y := V(C)$.

5.2.8 (b) Use induction on the number of blocks of G. If G is separable, consider a separation $\{G_1, G_2\}$ of G. Apply the induction hypothesis to G_1 and G_2.

5.2.10 There is such a graph on four vertices.

5.2.11 How many endblocks are there in $G \setminus e$?

5.2.12 (a) Let G be an even graph. Proceed by induction on n. One may assume that G has no loops (why?). Consider a vertex v. Each partition into pairs of the edge cut $\partial(v)$ gives rise to an even graph on $n-1$ vertices and $m - \frac{1}{2}d(v)$ edges by splitting off these pairs.

Section 5.3

5.3.2 Apply Exercise 5.2.11.

5.3.3 Show that no odd cycle has an ear.

5.3.4 Observe that the statement is true if G is a cycle. Use induction on the number of ears in an ear decomposition of G.

5.3.7 Consider the final ear P in an ear decomposition of G and proceed by induction on the number of ears. There are various cases to examine, depending on the number of internal vertices of P in $X \cup Y$.

Section 5.4

5.4.4 Proceed along the lines of Exercise 5.3.8, using Theorem 5.13.

Section 6.1

6.1.1 Use induction on $\ell(v)$.

6.1.7 If T is not a DFS-tree (rooted at x), there is a *cross edge* in G (an edge joining two unrelated vertices of T). Using this edge, find a spanning x-tree which contradicts the choice of T.

6.1.9 (a) Set $\rho(T_v) := \sum_{u \in V \setminus \{v\}} d(u,v)$. Show that $\sigma(T_v) \leq (n-1)\rho(T_v)$ and that $\sum_{v \in V} \rho(T_v) = 2\sigma(G)$.

6.1.11 Let P be a Hamilton path of G. Show that P extends to a Hamilton cycle C of G and that if xy is a chord of C, then x^+y^+ and x^-y^- are also chords of C. If the length of xCy is at least four, show further that $x^{++}y$ and x^+y^- are chords of C as well. Now consider whether or not C has a chord xy with xCy of length two.

Section 6.2

6.2.1 Suppose that T_1 and T_2 are two optimal trees in G. Apply Exercise 4.3.2, choosing the edge e appropriately.

6.2.5 Use the fact that $\log \alpha\beta = \log \alpha + \log \beta$.

6.2.6 (a) Let T_2 be a spanning tree whose largest edge-weight is smaller than the largest edge-weight in T. Apply Exercise 4.3.2 with $T_1 = T$, choosing the edge e appropriately.

Section 6.3

6.3.7 Return the vertices in an appropriate order depending on the function l.

6.3.8 Return the vertices in an appropriate order depending on the function f.

6.3.13 (b) Consider a DFS-tree in G which is rooted at one end of the path P and contains P. Orient G accordingly.

Section 7.1

7.1.1 (a) Observe that both $\{\partial^+(v) : v \in V\}$ and $\{\partial^-(v) : v \in V\}$ are partitions of the arc set A.

7.1.2 (a) Arcs with exactly one end in X appear in precisely one of the two sets $\cup\{\partial^+(v) : v \in X\}$ and $\cup\{\partial^-(v) : v \in X\}$, whereas those with their two ends in X appear in both of these sets.

7.1.4 Use Theorem 3.6.

Section 7.2

7.2.3 By appropriate scaling, all capacities may be taken to be integers.

7.2.4 If f is a nonzero flow, consider the set of f-positive arcs.

Section 7.3

7.3.1 (a) Use Exercise 7.1.2.

Section 8.1

8.1.3 See Exercise 5.2.7.

8.1.4 Start with a mapping between the centres of the two trees (see Exercise 4.1.8b). Now use a recursive algorithm for deciding whether two rooted trees are isomorphic.

Section 8.2

8.2.1 Use the vertex-splitting operation illustrated in Figure 8.2 for reducing Problem 8.5 to Problem 7.10.

Section 8.3

8.3.5 (a) Use a vertex-splitting operation similar to the one illustrated in Figure 8.2, inserting a path of length two rather than an arc.

8.3.9 There are three literals in each clause of f, and thus eight triples of possible values, seven of which satisfy the clause. Let each part V_i of G correspond to a clause of f, and the seven vertices in V_i correspond to these seven triples of values.

8.3.10 (a) One may assume that there is no clause of the form $x \vee \overline{x}$. Form a graph $G = (V, E)$ whose vertex set V is the set of literals of f, there being an edge xy for each clause $x \vee y$ of f, and one for each pair $\{x, \overline{x}\}$ of literals contained in V. Call the latter set of edges M.

Starting with an arbitrary vertex as root, grow a tree T which includes all vertices of G reachable from the root by paths whose edges are alternately in $E \setminus M$ and M. Colour the vertices of T alternately red and blue, the root being red. If $V(T) \neq V(G)$, grow another such tree in $G - V(T)$, and continue in this way until a spanning forest of G has been constructed. Consider the truth assignment in which each red vertex is assigned the value 0 and each blue vertex the value 1.

Section 8.4

8.4.3 Find a polynomial reduction from HAMILTON CYCLE.

Section 8.6

8.6.7 To obtain $D(x, y)$ from $G[X, Y]$, adjoin two new vertices x and y, join x to all vertices in X and y to all vertices in Y, and orient the resulting graph appropriately.

Section 9.1

9.1.2 Observe that any vertex cut of $G \vee H$ must include either $V(G)$ or $V(H)$.

9.1.3 (a) If G is not complete, consider a vertex cut S of G and a smallest component of $G - S$. How large can the degree of a vertex in such a component be?

9.1.4 If G is not complete, consider a vertex cut S of G and a smallest component of $G - S$. How large can the degree of a vertex in such a component be?

9.1.6 Consider the edges of a longest cycle of G.

9.1.7 Observe that every loop is deletable, and that each member of a set of multiple edges is deletable. Now use Exercise 5.3.2. Note that by a contractible edge is meant an edge e such that G / e is 2-connected, not one such that $\kappa(G / e) = \kappa(G)$.

9.1.8 Let S be a minimal vertex cut of G. Show that every vertex of S is adjacent to at least one vertex of each component of $G - S$. Deduce that there are two vertices in distinct components of $G - S$ whose distance in G is two.

9.1.10 (a)

(i) Use induction on m. Consider two cases, according to whether or not G / e is minimally 2-connected, where $e \in E$. In the latter case, by using the fact that $(G / e) \setminus f = (G \setminus f) / e$ for any edge f of G / e, show that e is incident to a vertex of degree two.

(ii) Proceed as in (i). If G / e is not minimally 2-connected for any $e \in E$, show that every edge of G is incident to a vertex of degree two, and apply a counting argument.

9.1.13 Use induction on the length of P.

Section 9.2

9.2.2 Consider a shortest odd cycle C together with a 3-fan from a vertex not on C to $V(C)$.

9.2.4 Find a 5-connected plane graph G whose outer face is of degree at least four, and place the four vertices x_1, y_1, x_2, y_2 in an appropriate order on the boundary of this face.

Section 9.3

9.3.3 Let $\partial(X)$ be a minimum edge cut of G. Use the fact that G is of diameter two to deduce that either $d(X) \geq |X|$ or $d(X) \geq |V \setminus X|$. Without loss of generality (why?) assume the former. Now apply Theorem 2.9 and Exercise 9.3.2a to deduce that $d(X) = \delta$.

9.3.4 Let $\partial(X)$ be a minimum edge cut. Without loss of generality (why?) it may be assumed that $|X| \leq n/2$. Show that $|X| \leq \delta$. Now use Theorem 2.9 and Exercise 9.3.2a to deduce that $d(X) = \delta$.

9.3.5 Consider the cases $\kappa = 0, 1, 2, 3$ separately.

9.3.12 Let $G := G(x, y)$ be a graph with two specified vertices x and y. Obtain G' from G by adding two new vertices x' and y' and joining x' to x and y' to y. Now let $H := H(u, v)$ denote the line graph of G', where u and v correspond to the edges xx' and yy' of G', respectively. Show that the maximum number of edge-disjoint xy-paths in G is equal to the maximum number of internally-disjoint uv-paths in H.

Section 9.4

9.4.7 If G / e is not 3-connected, denote by H the graph of minimum degree at least three of which $G \setminus e$ is a subdivision. Show that any two distinct vertices of H are connected by three internally disjoint paths.

9.4.8 There is an error in the statement of this exercise: 'any edge' should be replaced by 'some edge'. C. Thomassen showed how this theorem (Tutte's Wheel Theorem) can be derived from his Theorem 9.10. Assume that G is minimally 3-connected: $\kappa(G \setminus e) = 2$ for all $e \in E$. By Theorem 9.10, G contains an edge e such that G / e is 3-connected. Suppose that G / e is not simple, so that e is contained in a triangle. Show first of all that at least two vertices of that triangle have degree three. Thus G contains a path of length one whose vertices (i) are of degree three and (ii) have a common neighbour. Now consider a longest path in G with these two properties. Show that the ends of this path are adjacent, and that G has no other vertices. Conclude that G is a wheel.

9.4.9 (a) Show that no graph in $\mathcal{G}$ has two disjoint cycles, but that any simple graph obtained from a member of $\mathcal{G}$ by adding an edge or splitting a vertex of degree greater than three is either a 3-connected graph with two disjoint cycles or another member of $\mathcal{G}$. Prove the required assertion by induction on the number of edges, using Exercise 9.4.8.

Section 9.5

9.5.3 For $k = 8$, an example can be found amongst the figures in Chapter 14. Generalize this example to obtain one for all $k \geq 5$.

9.5.6 Let x and y be the two vertices of odd degree in G. Consider a balanced orientation of $G + xy$.

9.5.8 (b) If G is not minimally $2k$-edge-connected, an edge may be deleted from G to obtain a $2k$-edge-connected graph of smaller size. On the other hand, if G is minimally $2k$-edge-connected, it has a vertex of degree $2k$, by Exercise 9.3.15. Now use Theorem 9.16.

Section 9.6

9.6.3

(a) (i) Show that $d(X)$ is odd or even according to whether X contains an odd or an even number of vertices of odd degree. In particular, an odd edge cut separates some pair of vertices of odd degree.

(ii) Suppose that $d(X)$ is even. Using parity arguments and submodularity, show that there is a smallest odd edge cut $\partial(Y)$ which does not cross $\partial(X)$.

(b) By (a), a smallest odd edge cut among the $n - 1$ cuts determined by the Gomory-Hu algorithm is a smallest odd edge cut of G.

Section 9.7

9.7.2 Use induction on the length of the sequence of cliques in the decomposition.

9.7.3 Use induction on the order of the graph.

9.7.4 (a) Consider two disjoint subtrees T_i and T_j of a tree T, each of which intersects a third subtree T_k. Show that T_k contains the path in T connecting T_i and T_j.

Section 10.1

10.1.5 First, show that G can be assumed 3-connected. Next, apply Theorem 9.6 to obtain a nonspanning cycle C of length at least three. Finally, apply the Fan Lemma (9.5).

10.1.6 Count the degrees and the number of vertices. Do you recognize this graph?

10.1.7 Place the vertices along the spine.

10.1.8 (c) To show that the Petersen graph has crossing number at least two, use the fact that the graph is edge-transitive and appeal to Exercise 10.1.3(a).

10.1.9 Consider an optimal drawing $\widetilde{G}$ of $G := K_{n+1}$, one with exactly $\mathrm{cr}(K_{n+1})$ crossings. Compute the average number of crossings induced in a K_n-subgraph of $\widetilde{G}$.

10.1.10 Consider a pair of crossing edges in an optimal drawing of the graph.

10.1.12 Consider the line segment joining two distinct points (p, p^2, p^3) and (q, q^2, q^3) of the given curve. Let (x, y, z) be a generic third point on this line segment. Show that $x^2 - y \neq 0$. Deduce that no three points of the curve are collinear. Now compute $(xz - y^2)/(x^2 - y)$. Deduce that the line segments joining (p, p^2, p^3) to (q, q^2, q^3) and (r, r^2, r^3) to (s, s^2, s^3), where p, q, r, s are distinct, can meet only if $pq = rs$. Choose the points at which to place the vertices of the graph accordingly.

Section 10.2

10.2.4 Use Exercise 10.2.2.

10.2.11 Use Exercise 10.2.12a.

10.2.15 Use induction on $v(H)$, the initial case being when T is a star and H a wheel. If T is not a star, consider a vertex of T all of whose neighbours but one are leaves.

Section 10.3

10.3.1 How many edges must be deleted from G in order to obtain a planar graph?

10.3.2 Look at the proof of Corollary 10.21.

10.3.3 How many edges are there in a spanning tree?

10.3.4

(a) How many edges has K_{11}?
(b) There is a self-complementary plane graph on eight vertices which is also self-dual.

10.3.5 (a) Use Euler's Formula.

10.3.8 (a)(i) Use Euler's Formula.

10.3.9 (a) Form an appropriate planar graph.

10.3.10 Form an appropriate planar graph.

Section 10.4

10.4.1 Use Proposition 10.5.

10.4.4 Apply Exercise 9.4.6 and Theorem 10.28.

Section 10.5

10.5.3 (b) Use induction on the number of contractions needed to obtain K_5 as a minor.

10.5.5 Refine the proof of Theorem 10.35.

10.5.6 Place the vertices along the positive and negative x- and y-axes.

Section 11.1

11.1.2 Use planar duality.

11.1.3 Take $\mathbb{Z}_2 \times \mathbb{Z}_2$ as the set of colours.

11.1.4 Take $\mathbb{Z}_2 \times \mathbb{Z}_2$ as the set of colours.

11.1.7 (a) Apply Exercise 11.1.6d.

11.1.8

(a) Let xy be an edge of a triangulation $G = (V, E)$. Show that there is a unique 3-vertex colouring $c : V \to \mathbb{Z}_3$ with $c(x) = 0$ and $c(y) = 1$. (See also Theorem 21.5.)

(b) Show that every plane graph G has a supergraph H which is an eulerian plane triangulation. (Given a 3-vertex-colouring c of G, one can in fact find an eulerian plane triangulation H with a 3-vertex-colouring c' such that $G \subseteq H$ and c is the restriction of c' to $V(G)$.)

Section 12.2

12.2.4 See the remark following the proof of Turán's Theorem (12.7).

12.2.5 See the proof of Turán's Theorem (12.7).

12.2.7 (a) Assume that G contains no triangle. Choose a shortest odd cycle C in G. Show that each vertex in $V(G)\setminus V(C)$ is joined to at most two vertices of C. Apply Mantel's Theorem (Exercise 2.1.16) to $G - V(C)$, and obtain a contradiction.

12.2.8 (b) Apply the Cauchy-Schwartz Inequality.

12.2.9 (a) Appealing to Exercise 12.2.7, recursively delete edges in triangles until the resulting graph either (i) is bipartite , or (ii) has at most $\lfloor \frac{1}{4}(n-1)^2 \rfloor + 1$ edges. In case (i), conclude by showing that if G is a simple bipartite graph, then any two vertices in the same part have at least $m - \frac{1}{4}(n-1)^2$ common neighbours.
Alternatively, if each edge of G lies in a triangle, show that $t(G) > n^2/12$ and hence that $t(G) \geq \lfloor n/2 \rfloor$. If some edge $e = xy$ lies in no triangle, apply induction to $G - \{x, y\}$ using (a); observe, also, that $G - \{x, y\}$ is not bipartite, and hence that either x or y belongs to a triangle.

12.2.10 (a) A graph contains $K_{2,k}$ if and only if it has k vertices with a pair of common neighbours. Apply the Pigeonhole Principle.

Section 12.3

12.3.7

(b) Show that any simple graph G with $\delta \geq m-1$ contains every tree on m vertices.
(c) Use induction on n and the fact that any simple graph with $\delta \geq m-1$ contains every tree on m vertices.

12.3.8 (a)(ii) Let $(A_1, A_2, \ldots, A_n)$ be a partition of $[1, s_n - 1]$. Set $B_i := \{a + 2s_n - 1 : a \in A_i\}$ and $C_i := A_i \cup B_i$, $1 \leq i \leq n$, and $C_{n+1} := [s_n, 2s_n - 1]$. Show that $(C_1, C_2, \ldots, C_n)$ is a partition of $[1, 3s_n - 2]$ with no solution.

12.3.11 A 2-edge colouring of the countably infinite complete graph on a set $U = \{u_1, u_2, ...\}$ is said to be *canonical* if, for any fixed $i \geq 1$, all edges $u_i u_j$, where $j > i$, have the same colour. Firstly, show that given any 2-edge colouring c of a countably infinite complete graph K with $V(K) = \{v_1, v_2, ...\}$, there exists a countably infinite subset $U = \{u_1, u_2, ...\}$ of V such that the restriction of c to the subgraph $K[U]$ of K is canonical. Then show that every canonically 2-edge coloured countably infinite complete graph contains a monochromatic infinite complete subgraph.

12.3.12 Let z be a vertex on the boundary of the convex hull of V. Each line L which passes through z but through no other vertex of G partitions $V \setminus \{z\}$ into two subsets, X_L and Y_L. Consider all such lines L for which X_L and Y_L are nonempty, and apply induction to $G[X_L \cup \{z\}]$, $G[Y_L \cup \{z\}]$ and $G[V \setminus \{z\}]$.

Section 13.1

13.1.1 *Correction*: $A(S)$ and $A(T)$ should be replaced by A_S and A_T.

13.1.3

(a) Apply the Inclusion–Exclusion Formula (2.3), noting that $P(\overline{A_i}) = 1 - P(A_i)$.
(b) Use (a).

13.1.4 Use induction on $|I \setminus S|$.

Section 13.2

13.2.2 (a) Use induction on $|I|$.

13.2.4 Complete graphs have many crossings.

13.2.8 (a) What is the probability that a given linear ordering of the vertices is a directed Hamilton path?

13.2.11 (a)(i) Use the inequalities of Exercise 13.2.1.

13.2.18 (b) Let G and H be two such graphs. It may be assumed that $V(G) = V(H) = \mathbb{N}$. Construct an isomorphism $\theta : G \to H$ inductively, one vertex at a time. Initially, define $\theta(1) = 1$. Suppose that θ has been defined on a finite subset S of $\mathbb{N}$. Let i be the least integer not in S. Using the given property, show that there is a vertex $j \in \mathbb{N} \setminus T$, where $T := \theta(S)$, such that θ may be extended to $S \cup \{i\}$. Show, likewise, that θ^{-1} may be extended from $T \cup \{j\}$ to $T \cup \{j, k\}$, where k is the least integer not in $T \cup \{j\}$.

Section 13.3

13.3.5 Consider a random 2-colouring of V.

13.3.6 Consider a random tournament.

Section 13.4

13.4.2
(b) Apply Markov's Inequality.
(c) Apply Chebyshev's Inequality.

13.4.3
(a) Apply Cayley's Formula (Theorem 4.8).
(b) Use Exercise 13.4.2.
(c) Apply Chebyshev's Inequality.

Section 14.1

14.1.1 What is the stability number of this graph? Use symmetry to determine all the maximum stable sets. Can the vertex set of the graph be expressed as the union of three of them?
Alternatively, in any 3-colouring of a 5-cycle, one colour is assigned to just one vertex, and the remaining two colours are assigned to two vertices each. Choose a 5-cycle in the graph and show that no 3-colouring of it can be extended to a 3-colouring of the entire graph. Use symmetry to limit the number of cases to be checked.

14.1.2 If G is not 2-connected, consider a cutvertex of G and apply induction on the number of blocks.

14.1.3 (a) If some colour includes no such vertex, show that this colour is not needed.

14.1.4 If S_i is the set of colours used for G_i, $i = 1, 2$, find a proper colouring of G by the set $S_1 \times S_2$.

14.1.5 (c) Use induction on n.

14.1.6 For the lower bound, apply one of the inequalities in Section 14.1. For the upper bound, observe that the m-sets containing a fixed element of the n-set S form a stable set in $KG_{m,n}$, as do the m-sets contained in a $(2m-1)$-subset of S.

14.1.7 Apply Exercise 1.1.11.

14.1.8 This should be classified as a hard exercise.

(a) Consider an odd cycle C of G. How many colours are needed for C and how many for $G - C$?
(b) Use induction on n. Show first that G contains a triangle. Consider a vertex v. If $N(v)$ is a stable set, contract $\partial(v)$ and apply induction. Now consider the vertices v_1, v_2, v_3 of a triangle T and their neighbour sets V_1, V_2, V_3 in $G - T$. Can there be edges in both $G[V_i \setminus V_j]$ and $G[V_j \setminus V_i]$? If there is no edge in $G[V_i \cap V_2 \cap V_3]$, find a 4-colouring of G.

14.1.9 Order the vertices of G according to their colour in a proper χ-colouring of G.

14.1.10

(a) Order the vertices of G according to their degree.
(b) Let $k := \max\{\min\{d_i + 1, i\} : 1 \leq i \leq n\}$. Show that $d_k \geq k - 1$, and apply Theorem 1.1.

14.1.11 (a) Use two of the inequalities in Section 14.1.

14.1.12 One may assume that G is connected (why?). Consider a DFS-tree T in G. Recall that the vertices on any single level form a stable set.

14.1.13 (b) Obtain a lower bound on $\chi(C_5[K_3])$ by considering $\alpha(C_5[K_3])$.

14.1.14 (a) Obtain a lower bound on $\chi(C_5[K_n])$ by considering $\alpha(C_5[K_n])$.

14.1.15 (b) Consider a kernel S of D, colour the vertices of S, delete S, and use induction.

14.1.16 (a) Let D_1 and D_2 be spanning subdigraphs of D, where the arcs of D_1 are the arcs (u, v) of D for which $f(u) \leq f(v)$, and the arcs of D_2 are the arcs (u, v) for which $f(u) > f(v)$. Show that either $\chi(D_1) \geq m + 1$ or $\chi(D_2) \geq l + 1$, and apply the Gallai-Roy Theorem (14.5).

14.1.17 Consider a proper χ-colouring of G, and orient G accordingly.

14.1.23 Consider a proper χ-colouring of G, and a random partition of its colour-classes into two equal, or almost equal, subsets.
Alternatively, consider all such partitions and apply an averaging argument.

14.1.26 Observe that the neighbours of any vertex can be properly coloured in two colours. Iteratively select a vertex of sufficiently high degree k (to be determined) in the subgraph of G induced by the as yet uncoloured vertices, and assign such a colouring to its neighbours using two new colours each time. Now colour the remaining subgraph (whose maximum degree is less than k) by the greedy heuristic, again using fresh colours.

14.1.28

(a) Show that a spanning branching forest F in D which maximises $\sum_i in_i$, where n_i is the number of vertices at level i in F, has the desired property.
(c) For $0 \leq j \leq l - 1$, set $X_j := \cup_{r \geq 0} S_{k+j+rl}$. Show that some X_j is not a stable set, and consider the branching or branchings of F containing x and y, where $(x, y) \in D[X_j]$.

14.1.29 For G countable, let $V = \{v_0, v_1, \ldots\}$ and set $G_i := G[\{v_0, v_1, \ldots, v_i\}]$, $i \in \mathbb{N}$. Denote by $\mathcal{C}_i$ the set of proper colourings of G_i in the colours $1, 2, \ldots, k$. Define a graph with vertex set $\cup\{\mathcal{C}_i : i \in \mathbb{N}\}$, where $c_i \in \mathcal{C}_i$ and $c_j \in \mathcal{C}_j$ are adjacent if $j = i + 1$ and c_i is the restriction of c_j to $\{v_0, v_1, \ldots, v_i\}$. Now apply König's Lemma (Exercise 4.1.21). (For G uncountable, see Diestel (2005), p. 201.)

Section 14.2

14.2.1 Consider a critical subgraph.

14.2.3 This exercise is incorrect. The Chvtal graph is *not* 4-critical. A 4-critical subgraph of the graph may be obtained by deleting two appropriately chosen edges.

14.2.10 Use Exercise 14.2.9.

14.2.12 (a) Use Theorems 14.10 and 14.7.

14.2.16 Let $\mathcal{C} := (V_1, V_2, \ldots, V_k)$ be a k-colouring of G with as few singleton colour classes as possible, and let $\mathcal{C}' := (V_1', V_2', \ldots, V_l')$ be a colouring of G in which each colour class contains at least two vertices. Form a bipartite graph $F := F[X, Y]$, where $X := \{V_1, V_2, \ldots, V_k\}$ and $Y := \{V_1', V_2', \ldots, V_l'\}$, with V_i adjacent to V_j' if and only if $V_i \cap V_j' \neq \emptyset$. Assume that $|V_1| = 1$. Show that V_1 is connected in F to some V_i with $|V_i| \geq 3$, and modify $\mathcal{C}$ accordingly.

14.2.18 Apply Theorems 14.7 and 2.4.

Section 14.3

14.3.1 The fact that G_k is k-chromatic is established by Theorem 14.12. There are three kinds of edges in G_k. For those in G_{k-1}, use induction.

14.3.2 Colour the set of all vertices $\{i, j\}$ of SG_n such that $1 \leq i \leq n/2$ and $n/2 < j \leq n$. Now apply induction to colour the remaining vertices.

14.3.3 If H is k-colourable, there is an n-subset of S all of whose elements receive the same colour. Consider the corresponding copy of G to obtain a contradiction.

Section 14.4

14.4.2 To show that $\overline{C_{2k+1}}$ is imperfect, use an inequality from Section 14.1.

14.4.3 (a) Use Theorem 9.22 and induction on n.

14.4.4 Use induction on the order of the graph.

14.4.7 This should not be classified as a hard exercise. Simply apply Theorem 14.14.

14.4.8 If G is not perfect, it has a minimally imperfect induced subgraph. Give a succinct certificate to recognize such a subgraph.

Section 14.5

14.5.3 This should be classified as a hard exercise.
It may be assumed that not all the lists are identical (why?). If G is not regular, find an appropriate orientation of G and apply Theorem 14.20. If G is regular, consider an endblock B of G. If B contains adjacent vertices with distinct lists, one of them is not a cut vertex (why?). Assign it a colour not in the other's list, delete this colour (if present) from the lists of its neighbours, delete the vertex from G, find an appropriate orientation of the resulting graph, and apply Theorem 14.20. If all the vertices of B have the same list, obtain a Δ-list-colouring of G by colouring separately the block B and the union of the remaining blocks of G.

14.5.5 Denote the bipartition of K_{n,n^n} by (X, Y), where $|X| = n$ and $|Y| = n^n$. To show that this graph is not n-list-colourable, assign disjoint n-sets to the vertices of X and appropriate lists to the vertices of Y.

14.5.7 Consider any choice of colours from the lists in one part of the bipartition of $K_{n,n}$. Let T be the set of colours chosen. Show that $|T| \geq k$.

14.5.8

(a) Use induction on k. Consider two cases, according to whether or not there exist two consecutive vertices on the k-path whose lists are distinct.
(b) Use induction on n. To show that a (bipartite) theta graph with at least two of its three paths of length three or more is not 2-list colourable, assign the same list to all the vertices on the third path and appropriate lists to the other two paths. To show that a 2-list-colourable graph with no vertices of degree less than two and at least two vertices of degree at least three is a theta graph, find a list assignment for an even cycle C such that a given vertex of C is assigned the same colour in every compatible list colouring. Deduce that any two cycles in a 2-list-colourable graph have at least two vertices in common.

14.5.9 For $v \in V$, let $c(v)$ be a colour chosen at random from $L(v)$. For $e \in E$, consider the 'bad' event A_e that the ends of e receive the same colour.

14.5.10 (a) Show that an interval graph G may be regarded as the intersection graph of n subpaths $P_1, P_2, \ldots, P_n$ of a path P, where P_n is simply the terminal vertex of P. Apply induction on n to obtain the desired orientation of G.

Section 14.6

14.6.2 (b) Observe that the subgraph of D induced by $A(D) \setminus A(D')$ is an even digraph, and apply Exercise 2.4.2.

14.6.5 (a) The converse of an eulerian orientation of G is eulerian. Show that the two have the same sign.

14.6.6 (a) Apply Corollary 14.21, and appeal to Exercise 21.4.5.

Section 14.7

14.7.3 (a) Apply induction, using the recursion $P(G,x) = P(G \setminus e, x) - P(G / e, x)$.

14.7.4 (b) Compute the coefficient of x^{n-1} in $P(G,x)$, and apply Exercise 14.7.3(a).

14.7.5 Use the recursion $P(G,x) = P(G \setminus e, x) - P(G / e, x)$ and Exercise 14.7.4a.

14.7.9 Prove that $(-1)^{n-1}P(G,x) > 0$ when G is connected and $0 < x < 1$. Show this for a tree, then use the recursion $P(G,x) = P(G \setminus e, x) - P(G / e, x)$.

14.7.10 Consider the expansion of $P(G,x)$ in terms of chromatic polynomials of complete graphs.

14.7.11 Denote by $f(G)$ the number of acyclic orientations of G. Show that $f(G) = f(G \setminus e) + f(G / e)$. Now proceed by induction on m.

14.7.12 Consider the set of all k^n functions from $V(G)$ into $\{1, 2, \ldots, k\}$. For $e \in E$, denote by A_e the set of such functions which assign the same value to both ends of e. For $S \subseteq E$, how many functions assign the same value to both ends of e

for all $e \in S$? Express the set of all proper vertex colourings of G in terms of the sets A_e, and apply the Inclusion-Exclusion Formula (2.3).

Section 15.1

15.1.1 Look for an embedding in which all faces have degree five or six.

Section 15.3

15.3.1 (a) Check the theorems in Chapter 14.

Section 15.4

15.4.2 (a) If $\delta \geq 3$, use Exercise 10.1.5. If there is a vertex of degree less than three, delete it and use induction.

15.4.3 See Exercise 14.1.13.

15.4.6 Construct a maximal stable set S in G by means of a greedy algorithm, adding at each stage a vertex at distance two from the current stable set. Show that there is a tree T in G containing S with at most $2|S| - 1$ vertices.

15.4.7 One implication is easy. For the other, consider a largest induced subgraph whose vertex set can be partitioned into stable sets of cardinality two, and make use of the ceiling function $\lceil n/2 \rceil$.

Section 16.1

16.1.3 Show that $\overline{G}$ has no stable sets of size greater than two. Use this observation to establish a bijection between matchings in G and colourings of $\overline{G}$.

16.1.7

(a) Apply Theorem 2.9 to the spanning 1-regular subgraph $G[M]$ of G.
(b) Let S denote the vertex set of a 5-cycle of the Petersen graph. Show that $|M \cap \partial(S)| \neq 3$. Use part (a) and this property of the Petersen graph.

16.1.8

(a) Apply Exercise 16.1.7a.
(b) Look at the three cut edges.
(c) Construct a $(2k+1)$-regular simple graph in which there is a (stable) set of $2k-1$ vertices whose deletion results in a graph with $2k+1$ components, each of which is odd. (One may construct such a graph whose connectivity is $2k-1$.)

16.1.9 Consider the subgraph $G[M \triangle M^*]$.

16.1.10 Consider the sum of the lengths of the edges of the matching.

16.1.11 If G has a perfect matching M, show that the second player has a response to every move of the first player. If G has no perfect matching, consider a maximum matching M of G. By appealing to Berge's Theorem (16.3), devise a winning strategy for the first player in which he starts by picking a vertex not covered by M.

16.1.12 Let M be a maximum matching of G. If $|M| < \delta$, show that any two vertices not covered by M are connected by an M-augmenting path of length three, and derive a contradiction.

16.1.13

(a) *Correction.* This part should read:
Show that G has no M-alternating cycle, and that the first and last edges of every *maximal* M-alternating path belong to M.
(b) Show that the two ends of any maximal M-alternating path have degree one in G.
(c) There is an example on six vertices.

16.1.14

(a) Use Berge's Theorem (16.3).
(b) The hypothesis implies that every vertex is incident with a link. Use part (a).
(c) Let M_A be a maximum matching of G which (i) covers every vertex in A and (ii) subject to (i), covers as many vertices of B as possible. Define M_B similarly. Assume that there is a vertex u in A which is not covered by M_B. Derive a contradiction by considering the component of $G[M_A \triangle M_B]$ containing the vertex u.

16.1.15 (b) Let $G[X, Y]$ be a bipartite graph and let xy be an edge of G with $x \in X$ and $y \in Y$. Suppose that neither x nor y is essential. Then there exist maximum matchings M_x and M_y of G such that M_x covers x but not y, and M_y covers y but not x. Show that no component of $G[M_x \triangle M_y]$ contains both x and y. Let P_x and P_y be the components of $G[M_x \triangle M_y]$ containing x and y, respectively. Obtain a contradiction by considering the subgraph $(P_x \cup P_y) + xy$. (Thus, in a bipartite graph, at least one end of each edge is essential.)

16.1.16 Let G denote the complement of the comparability graph associated with the partially ordered set $(\{1, 2, \ldots, n\}, \prec)$. (Thus ij is an edge of G if and only if neither $i \prec j$ nor $j \prec i$.) By induction on n, show that (i) the minimum of $d_1 + d_2$ is $n - \alpha'$ and (ii) there exists a maximum matching M of G and a feasible schedule for processing all the jobs in which the jobs corresponding to the vertices not covered by M are processed on the first machine and the pairs of jobs corresponding to the edges of M are processed on the second machine.

Section 16.2

16.2.4 Consider $G - v$, where v is an essential vertex.

16.2.7 (a) Consider the set $S := X \setminus K$, where K is a minimal covering of G.

16.2.10 How many edges can be incident to $k - 1$ vertices of G?

16.2.11 (a) Consider random proper k-colourings of $G[X]$ and $G[Y]$. Call an edge $e \in [X, Y]$ *bad* if the two ends of e receive the same colour. What is the probability of this event? Deduce that the expected number of bad edges is less than one, and apply Markov's Inequality (13.4) with $t = 1$.

16.2.12 Let M be a maximum matching. For each vertex v not covered by M, pick an edge incident to v. and let F be the set of these edges. By considering the set $M \cup F$, deduce that $\alpha' + \beta' \leq n$. To show that $\alpha' + \beta' \geq n$, consider a minimum edge covering of G, and a maximum matching in the subgraph of G induced by that edge covering.

16.2.13 For $S \subseteq X$, consider the bipartite adjacency matrix of $G[S \cup N(S)]$. Show that the rows of this matrix are linearly independent over $GF(2)$ by considering the inner product of any sum of its rows with any single row of that sum. Deduce that this matrix has at least as many columns as rows, and apply Hall's Theorem (16.4).

16.2.14 Use Proposition 1.3.

16.2.15 Apply Hall's Theorem to a suitable bipartite graph constructed from G.

16.2.17 For $k < n$, one may assume that the union of any k of the cycles has at least $k + 1$ vertices (why?). Deduce that $(V(C_i) : 1 \leq i \leq n)$ has an SDR $(v_i : 1 \leq i \leq n)$, and consider suitable arcs $a_i \in A(C_i)$, $1 \leq i \leq n$.

16.2.19 Construct a bipartite graph $G[X, Y]$ in which X and Y are the sets of rows and columns, respectively, of $\mathbf{Q}$, and row i is joined to column j if and only if the entry q_{ij} is positive. Show that G has a perfect matching. Now use induction on the number of nonzero entries of $\mathbf{Q}$.

16.2.24

(a) Suppose that there is a proper subset of X which is tight. Using the fact that G is connected, find an edge which is not contained in a perfect matching and conclude that G is not matching-covered. Conversely, if G is not matching-covered, there exists an edge xy with $x \in X$ and $y \in Y$ such that $G - \{x, y\}$ does not have a perfect matching. Using Hall's Theorem (16.4), show that there is a proper subset of X which is tight.

(b) Similar to the second part of (a).

16.2.25 Use Exercise 16.2.24 and induction on the number of vertices.

16.2.27 First show that part (b) follows from part (a).

Prove part (a) by induction on n by proceeding as follows. Let x be the vertex in X of degree two, and let xy_1 and xy_2 be the two edges incident with x. Consider two cases depending on whether or not $y_1 = y_2$.

If $y_1 = y_2$, the two edges xy_1 and xy_2 are multiple edges. Apply the induction hypothesis to $H := G - \{x, y_1\}$.

If $y_1 \neq y_2$, consider the graph $H := (G - x)/\{y_1, y_2\}$. Denote the vertex of H resulting from identifying y_1 and y_2 by y. If one of y_1 and y_2 has degree two in G, the graph H is a cubic bipartite graph on $2n - 2$ vertices. Otherwise, vertex y has degree four in H. Let $e_1, e_2, e_3,$ and e_4 denote the four edges incident with y in H. Apply the induction hypothesis to $H \setminus e_i$, $1 \leq i \leq 4$.

Section 16.3

16.3.5 Let M be a maximum matching of G and let U denote the set of vertices not covered by M. Since v is essential, $\alpha'(G-v) = \alpha'(G)-1$, and hence v is covered by M. Let uv be the edge of M incident with v. Then observe that $M' := M \setminus \{uv\}$ is a maximum matching of $G-v$ and $U' := U \cup \{u\}$ is the set of vertices not covered by M' in $G - v$. Use the definition of a barrier (as applied to $G - v$) and the fact that $o(G - (B \cup \{v\})) = o(G - v - B)$ to deduce that $B \cup \{v\}$ is a barrier of G.

16.3.6 If G is a connected graph without essential vertices, then it follows from Lemma 16.10 that the empty set is a barrier of G. Thus, we only need to consider disconnected graphs. Observe that if $G_1, G_2, \ldots, G_k$ are the components of G, then every maximum matching of G is the union of maximum matchings of the G_i.

Section 16.4

16.4.1 If $o(G - v) = 1$ for all $v \in V$, consider the neighbour y of a leaf x and apply induction to $G - \{x, y\}$.

16.4.4 See the hint to Exercise 16.4.5.

16.4.5 More generally, let us consider the problem of finding necessary and sufficient conditions under which a graph G has a matching which covers all the vertices of a specified subset X of V. Obtain a graph H from G as follows:

- ▷ if n is odd, adjoin an isolated vertex to G;
- ▷ for each pair of nonadjacent vertices, both not in X, add an edge joining them.

1. Show that there is a matching in G covering all vertices in X if and only if H has a perfect matching.
2. By applying Tutte's Theorem (16.13), deduce that G has a matching covering all vertices in X if and only

$$o(G[X \setminus S]) \leq |S| \quad \text{for all} \quad S \subseteq V$$

3. Now suppose that $G[X]$ is bipartite. Strengthen the statement in (b) to show that there is a matching covering all vertices in X if and only if the number of isolated vertices of $G[X \setminus S]$ is at most $|S|$ for all $S \subseteq V$.

The above results may be adapted to solve this exercise, as well the previous exercise.

16.4.6 Let H be a maximal spanning supergraph of G whose matching number is the same as that of G. Use the proof technique of Theorem 16.13 to show that if U is the set of vertices of degree $n-1$ in H, then $H - U$ is a disjoint union of complete graphs.

16.4.7

(a) By hypothesis, G has a perfect matching. Therefore, for any barrier B of G, we have $o(G - B) = |B|$. Firstly, suppose that vertices x and y belong to some barrier B. To show that $G - \{x, y\}$ has no perfect matching, observe that $(G - \{x, y\}) - (B \setminus \{x, y\}) = G - B$. Conversely, suppose that x and y are two vertices such that $G - \{x, y\}$ has no perfect matching. Show that if B' is any barrier of $G - \{x, y\}$, then $B := B' \cup \{x, y\}$ is a barrier of G.

(b) Observe that an edge xy of G is contained in some perfect matching of G if and only if $G - \{x, y\}$ has a perfect matching.

16.4.8 By contradiction. Suppose that an edge xy is not in any perfect matching. By Exercise 16.4.7(b), there exists a barrier of G which contains both x and y. Let B be such a barrier and let $G_1, G_2, \ldots, G_{|B|}$ be the odd components of $G - B$. From the hypothesis that G is 3-regular and the fact that x and y are adjacent, deduce that $|\partial(B)| \leq 3|B| - 2$. Now, derive a contradiction using the hypothesis that G has no cut edges.

16.4.10

(a) Use Exercise 16.4.7.

(b) Use counting arguments similar to the ones employed in the proof of Petersen's Theorem (16.14) and Exercise 16.4.8.

16.4.11

(a) Consider a spanning tree T of G with the least number of leaves. If $n \geq 4$, show that T has a leaf x adjacent to a vertex y of degree two. Apply induction to $G - \{x, y\}$.
Alternatively, let G be a connected claw-free graph on an even number of vertices, and let $S \subseteq V$. Form a simple bipartite graph $H := H[S, T]$, where T is the set of odd components of $G - S$, there being an edge of H joining a vertex of S and an odd component of $G - S$ if and only there is such an edge in G. Show that $d_H(v) \leq 2$ for all $v \in S$. Using Exercise 16.3.2, deduce that $|T| \leq |S|$, and apply Tutte's Theorem.

16.4.13 Let G be a 2-connected graph with a perfect matching M, and let $X := \{x_1, x_2, \ldots, x_k\}$ be a maximal barrier of G. Then $G - X$ has k components, $G_1, G_2, \ldots, G_k$, each of which is hypomatchable. For $1 \leq i \leq k$, shrink $V(G_i)$ to a single vertex y_i and set $Y := \{y_1, y_2, \ldots, y_k\}$. Now delete the edges of G with both ends in X to obtain the bipartite graph $H[X, Y]$. Since, by Exercise 16.4.7, no edge with both ends in X can be in M, it follows that the restriction of M to $E(H)$ is a perfect matching of H. Using the fact that each G_i is hypomatchable, show that any perfect matching of H may be extended to a perfect matching of G. Now use Exercise 16.1.13.

16.4.14

(a) By Exercise 16.4.13, the graph either is disconnected or has a cut vertex. Consider its blocks, and apply induction.
(b) Construct such a graph recursively, adding two vertices at a time.
Alternatively, consider the (unique) connected simple graph on $2n$ vertices with $2n - 1$ distinct degrees.

16.4.16 (b) Let G be a $2k$-regular graph with $V = \{v_1, v_2, \ldots, v_n\}$; without loss of generality, assume that G is connected. Let W be an Euler tour in G. Form a bipartite graph $H[X, Y]$, where $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$, by joining x_i to y_j whenever v_i immediately precedes v_j on W. Show that H is 1-factorable. Deduce that G is 2-factorable.

16.4.17 Apply Exercise 16.4.16b.

16.4.18 Let G be a triangulation. Apply Petersen's Theorem (16.14) to G^*.

16.4.19 Let G be a 2-connected 3-regular graph on four or more vertices, and let M be a perfect matching of G. Then $F := G \setminus M$ is a 2-factor of G. Orient each cycle of F as a directed cycle, and use this orientation to associate a path of length three with each edge of M.

16.4.20 (a) If $d(v) \in L(v)$ for all $v \in V$, let $f(v) = d(v)$ for all $v \in V$. If $d(y) \notin L(y)$ for some vertex y, let $(x, y) \in A$. Set $G' := G \setminus xy$, $D' := D \setminus (x, y)$, $L'(x) := L(x) \setminus \{d(x)\}$ and $L'(v) := L(v)$, $v \in V \setminus \{x\}$, and apply induction.

Section 16.5

16.5.3 Show that any M-alternating path which enters T from $G - V(T)$ must enter via a non-matching edge incident with a blue vertex of T. Consider the successive vertices of this path.

16.5.6 Denote the k parts of G by $\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_k, y_k\}$. For $v \in V$, let $L(v)$ be a list of k colours assigned to v. The assertion is trivial for $k = 1$, so assume that $k \geq 2$. If $L(x_i) \cap L(y_i) = \emptyset$, $1 \leq i \leq k$, apply Hall's Theorem (16.4) to an appropriately defined bipartite graph $H[V, C]$, where $C := \cup\{L(v) : v \in V\}$. If not, proceed by induction on k.

Section 17.1

17.1.2 Find a decomposition of $K_{n,n}$ into perfect matchings.

17.1.3 How many edges can have the same colour?

17.1.5

(a) G_i is clearly connected and cubic. Can it have a cut edge?
(b) In a 3-edge-colouring of G, can e_1 and e_2 receive different colours? See Exercise 16.1.7a.

17.1.6 (a) In a 3-edge-colouring of G, can two edges of a 3-edge-cut receive the same colour? See Exercise 16.1.7a.

17.1.8 How many edges are needed to colour the edges of a Hamilton cycle?

17.1.10

(a) Can two classes be taught in the same period by the same teacher?
(b) Consider two periods in which the numbers of classes differ by at least two. Can the schedule be rearranged?

17.1.11 A regular bipartite graph of positive degree has the same number of vertices in each part (see Exercise 1.1.9).

17.1.14 Show that $\chi'(K_8) = 7$.

17.1.15 What is a vertex colouring of a line graph? What is a clique in a line graph? Express these concepts in terms of the root graph.

17.1.16 If intersecting triples have distinct colours, how many colours are needed to colour all triples?

17.1.17

(a) Place all the vertices but one in a circle, and the remaining one at the centre (as in the hint to Exercise 2.4.6a).
(b) See Exercise 17.1.3a.

17.1.18 Use the proof technique of Theorem 17.2.

17.1.19 Use induction on the number of vertices. Let G be a 3-edge-coloured complete graph with the given property, and v be a vertex of G. If some colour class of $G - v$ does not induce a connected spanning subgraph, look at the colours of the edges connecting two components in the subgraph induced by that colour. Can both of the other colours be present?

Section 17.2

17.2.1 (a) How many edges can there be of the same colour? (Compare with Exercise 17.1.3.)

17.2.2

(a) How many edges of the same colour can there be in H?
(b) Look at the constructions described in Exercise 17.2.1b.

17.2.3 This should be classified as a hard exercise.

17.2.4 (a) Consider a $(\Delta + 1)$-edge-colouring of G.

17.2.5 Find a suitable representation of the vertices of $V(P \,\Box\, K_3)$ by pairs of elements from $\{1, 2, 3, 4, 5\}$, and use it to determine a 5-edge-colouring of $P \,\Box\, K_3$ in the colours $1, 2, 3, 4, 5$.

17.2.8 (a) Adapt the proof technique of Lemma 17.3.

17.2.9 A forest has a vertex of degree zero or one. Bearing this in mind, apply Lemma 17.3.

17.2.10 (a) Can $G[M_i \cup M_j]$ be disconnected?

17.2.11 How are 3-edge-colourings and Hamilton cycles related in cubic graphs?

17.2.12 Show, by contradiction, that no regular self-complementary graph belongs to Class 1. The converse of this assertion is an unsolved problem. (See A.P. Wojda. A note on the colour class of a self complementary graph. *Discrete Math.* **213** (2000), 333–336.)

17.2.13 Use the proof technique of Lemma 17.3.

17.2.14 Use the proof technique of Lemma 17.3.

Section 17.3

17.3.3 (a) Use the result in Exercise 16.1.7a. (It implies that if (M_1, M_2, M_3) is a 3-edge-colouring of a cubic graph G, then, for any subset X of V, the parities of $|\partial(X) \cap M_i|$, $1 \le i \le 3$, are all equal to the parity of $|\partial(X)|$.)

17.3.4 (a) Adapt the ideas in the proof of Tait's Theorem (11.4).

17.3.5 (a) Use the fact that if X is the vertex set of a 5-cycle of G_k, and if M' is any perfect matching, then $|M' \cap \partial(X)| = 1$ or 5 (see the hint to Exercise 16.1.7b).

Section 17.4

17.4.2 The graph H has a perfect matching M (why?). If H is triangle-free, show that the graph obtained by duplicating the edges of $H \setminus M$ is 5-edge-colourable (consider its line graph). If H has a triangle, contract it to a single vertex and use induction.

Section 18.1

18.1.3 Could a hypothetical Hamilton cycle in the Meredith graph include both of the edges joining two $K_{3,4}$-subgraphs?

18.1.4 (b) A nontrivial path has just two ends.

18.1.5 Why is the Herschel graph (Figure 18.1b) nonhamiltonian? Modify this graph.

18.1.7 (b) A Hamilton-connected graph is hamiltonian. The Petersen graph is nonhamiltonian.

18.1.8 Orient the two triangles of a triangular prism in the same sense.

18.1.10 In a path partition of H, where are the ends of the constituent paths to be found?

18.1.11 (b) Is a graph in which every vertex-deleted subgraph is traceable path-tough?

18.1.12 Consider an edge uv and an automorphism α of G such that $\alpha(u) = v$.

18.1.14 Use Exercises 18.1.2 and 18.1.13.

18.1.15 (b) Recall that the Herschel graph is bipartite.

18.1.16 Use Exercise 18.1.14.

18.1.17 Use the fact that the Petersen graph is hypohamiltonian (Exercise 18.1.16a).

18.1.18

(a) If G has no k-cycle, show that if $vx \in E$ then $vy \notin E$ for some $y \in V$. Deduce that $d(v) \leq (n-1)/2$.
(b) Let C be a Hamilton cycle of G with a specified sense of traversal. Show that there are consecutive vertices x, y of C such that $d(x) + d(y) \geq n + 1$. Deduce that $G - v$ is hamiltonian for some $v \in V$, and examine the two cases $d(v) \leq (n-1)/2$ and $d(v) \geq n/2$.

18.1.19 (b)

(i) Consider the cycle space of G.
(ii) Bound the number of cycles that include at most two chords of the Hamilton cycle. Obtain the sharper result: if $n \leq \binom{r+2}{2} + 1$, then $m \leq n + r - 1$.

18.1.20 If $D' := D - \{x, y\}$ is strong, apply Camion's Theorem and look at Exercise 18.1.18a. If not, y dominates a vertex y' in the initial component of D' and x is dominated by a vertex x' in the terminal component of D'. Show that there exist directed (y', x')-paths in D' of all lengths l', $1 \leq l' \leq n - 3$.

Section 18.2

18.2.2 Start with a simple nonhamiltonian 2-connected cubic planar graph.

18.2.3 (b) Show that $V(G_i) \setminus V(G_{i-1})$ is a stable set of G_i. Deduce that the circumference of G_i is at most twice the circumference of G_{i-1}, $i \geq 1$.

Section 18.3

18.3.2 If there are two different closures, consider the first edge added in the formation of one of them that is not an edge of the other. Derive a contradiction.

18.3.3 (a) Use Exercise 18.1.6a and Theorem 18.9.

18.3.5 (a) Let P be a longest path in G. If P has length $l < 2\delta$, show, using the proof technique of Theorem 18.9, that G has a cycle of length $l+1$. Now use the fact that G is connected to obtain a contradiction.

18.3.6 Construct a certain graph H from G, with $\alpha(H) = \alpha(G)$ and $\kappa(H) = \kappa(G) + 1$. Apply Theorem 18.10.

18.3.7 Assume false. Consider a cycle C through as many vertices of X as possible, and a path P from $X \setminus C$ to C. Find two vertices (one on P and one on C) whose degree sum is less than n.

18.3.8

(a) Apply Theorem 18.1.
(b) Apply Theorem 18.9.

18.3.9 Form a new graph H from G by adding edges between all pairs of vertices of X. Show that H is hamiltonian if and only if G is hamiltonian.

18.3.10

(a) Apply Corollary 18.8.
(b) If $m = \binom{n-1}{2} + 1$, when are G and $c(G)$ identical?

18.3.11

(a) Bound the minimum degree δ.
(b) A *ladder* is a $(2 \times n)$-grid. Modify such a graph.

18.3.13 Use induction on the length of Q.

18.3.15

(a) Set $P := v_0v_1v_2\ldots v_\ell$, where $v_0 := x$, $v_k := y_1$ and $v_\ell := y$, and define

$$X := \{v_i : xv_{i+1} \in E\} \quad \text{and} \quad Y := \{v_i : v_iy \in E\}$$

Consider the cases $X \cap Y \neq \emptyset$ and $X \cap Y = \emptyset$. In the latter case, show that C contains every vertex of $(X \cup Y \cup \{y\}) \setminus \{v_{k-1}\}$.
(b) (ii) Apply (b)(i) to the graph obtained by adding a set S of $k-1$ vertices and joining them to one another and to both x and y. Consider two cases, according to whether or not the resulting cycle meets S.

Section 18.4

18.4.1

(a) (i) Apply Smith's Theorem (18.13).
(ii) See Exercises 17.2.10 and 17.2.11.
(b) See Exercise 18.1.2.

18.4.5 Apply the Lollipop Lemma (18.11).

18.4.6

(a) This is incorrect. The theorem published by A. Kotzig (Aus der Theorie der endlichen regulären Graphen dritten und vierten Grades. Časopis Pěst. Mat. **82** (1957), 76–92) is somewhat weaker than stated, and Exercise 18.4.5a should read as follows:

Let G be a simple cubic graph. Show that the line graph of G admits a Hamilton decomposition if and only if G is hamiltonian.

A Hamilton cycle C in a graph G is a dominating eulerian subgraph of G, and thus gives rise to a Hamilton cycle in $L(G)$ (see Exercise 3.3.8). Show how to obtain from C two edge-disjoint Hamilton cycles in $L(G)$. (Orienting C as a directed cycle and assigning arbitrary orientations to the remaining edges of G might help.)

Conversely, consider a decomposition of L(G) into two Hamilton cycles. Each vertex of $L(G)$ corresponds to an edge of G, and there are two distinct ways in which a cycle can traverse a vertex of $L(G)$. Show that the edges of G which correspond to the vertices of $L(G)$ traversed by the two Hamilton cycles in one of these ways induce a Hamilton cycle in G.

(b) This part remains valid.

18.4.7 (b) The medial graph of a plane even graph is 2-face-colourable. Consider the two colour-classes of faces.

18.4.8 (a)(i) Use induction on the distance between e and f.
Alternatively, prove the result for graphs G of even order only by constructing an appropriate bipartite graph from G and arguing as in Exercise 18.4.2.

18.4.9 Consider first the case $k = 4$. Reduce the problem to 4-regular graphs.

18.4.10 Subdivide each edge of $C - S$ once.

18.4.11 (a) Use the proof technique of the Lollipop Lemma (18.11).

18.4.12 (a) Use the proof technique of the Lollipop Lemma (18.11).

Section 18.5

18.5.1 Consider a longest path P in G, and define X as in Pósa's Lemma (Theorem 18.19).

Section 19.1

Section 19.2

Section 19.3

Section 19.4

19.3.1 Let $D := D(x,y)$ be a digraph. Obtain the digraph $D' := D'(x,y)$ from D by adding k new arcs from y to v for each vertex $v \in V \setminus \{x,y\}$. Show that D has k arc-disjoint directed (x,y)-paths if and only if D' has k arc-disjoint x-branchings.

19.3.4 Adjoin a new vertex x to D and join it to each $v \in V$ by $k - d^-(v)$ arcs. Denote the resulting digraph by D'. Show that D' has k arc-disjoint spanning x-branchings.

Section 19.5

19.4.9

1. See Figure B.1.

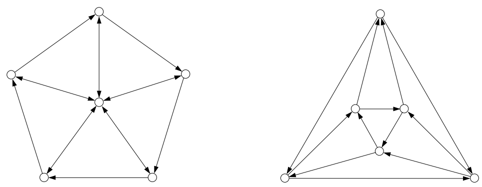

Fig. B.1. Two planar digraphs with $\nu = 1$ and $\tau = 2$

2. Clearly, we may assume that D is strong. Suppose the result false, and choose a counterexample D which is maximal, that is, such that the addition of any arc joining two nonconsecutive vertices on a face results in a digraph with two disjoint directed cycles. Let C be a facial directed cycle of D (such a cycle exists by Exercise 19.4.8). Show that, for any vertex v of C, there exists a facial directed cycle C_v of D which meets C only at v. Since $\nu = 1$, for $u, v \in V(C)$, $u \neq v$, the cycles C_u and C_v must have a vertex in common. Taking the planarity of D into consideration, analyse the possible intersection pattens amongst the cycles C_v, $v \in V(C)$, to obtain a contradiction.

19.4.11 Let $\mathcal{C}$ be a maximum laminar family of arc-disjoint directed cycles. Obtain a graph G whose vertex set is $\mathcal{C}$ by joining two elements C and C' by an edge if they have a vertex of D in common. Using the fact that D is 2-diregular and planar, deduce that G is planar. By the Four-Colour Theorem (11.2), G is 4-colourable. It follows that D has at least $\frac{1}{4}|\mathcal{C}|$ vertex-disjoint directed cycles. Thus $|\mathcal{C}| \leq 4k$.

By the Lucchesi-Younger Theorem (19.9), D has a set of $|\mathcal{C}|$ arcs whose deletion destroys all directed cycles. By selecting one end of each of these arcs, we obtain a set of at most $4k$ vertices whose deletion destroys all directed cycles.

20.4.1 (a) Let $\mathbf{Q}$ be a square submatrix of $\mathbf{K}$. Use induction on the order of $\mathbf{Q}$. If each column of $\mathbf{Q}$ contains two nonzero entries, then $\det \mathbf{Q} = 0$. Otherwise, expand $\det \mathbf{Q}$ about a column with exactly one nonzero entry, and apply the induction hypothesis.

20.4.3 Here is a proof of unimodularity.

Proof Let $\mathbf{Q}$ be a full square submatrix of $\mathbf{B}$ (one of order $n-1$). Suppose that $\mathbf{Q} = \mathbf{B}|_{\mathrm{T}_1}$. We may assume that T_1 is a spanning tree of D since, otherwise, $\det\mathbf{Q} = 0$ by Theorem 20.6. Let $\mathbf{B}_1$ denote the basis matrix of $\mathcal{B}$ corresponding to T_1. Then (Exercise 20.2.1(b))

$$(\mathbf{B}|_{\mathrm{T}_1})\mathbf{B}_1 = \mathbf{B}.$$

Restricting both sides to T, we obtain

$$(\mathbf{B}|_{\mathrm{T}_1})(\mathbf{B}_1|_{\mathrm{T}}) = \mathbf{B}|_{\mathrm{T}}.$$

Noting that $\mathbf{B}|_{\mathrm{T}}$ is an identity matrix, and taking determinants, we get

$$\det(\mathbf{B}|_{\mathrm{T}_1}) \det(\mathbf{B}_1|_{\mathrm{T}}) = \mathbf{1}. \tag{B.1}$$

Both determinants in (B.1), being determinants of integer matrices, are themselves integers. It follows that $\det(\mathbf{B}|_{\mathrm{T}_1}) = \pm\mathbf{1}$. □

20.5.1 Let W be the minimum power of an (x, y)-flow in D, and let f be a flow of power W. Being a flow, f satisfies Kirchhoff's Law (K1). To show that f is a current flow in D, we need to show that f also satisfies (K2); in other words, we need to show that f is a tension. Suppose that this is not the case. Then f is not orthogonal to the signed incidence vector $\mathbf{f}_C$ of some cycle C (because the tension space $\mathcal{B}$ is the orthogonal complement of the circulation space $\mathcal{C}$, which is generated by the vectors $\{\mathbf{f}_C\}$). Thus there is a cycle C, and a sense of traversal of C, such that $f^+(C) - f^-(C) < 0$. But then the flow $f + \epsilon f_C$, for ϵ sufficiently small, has power less than W, a contradiction.

To establish uniqueness, let f and g be two electrical currents of value I from x to y. Then $f - g$ is an electrical current of value zero, so is both a circulation in D and a tension in D. By the orthogonality of $\mathcal{C}$ and $\mathcal{B}$, it follows that $f - g = 0$, that is, $f = g$.

20.6.1 See Figure B.2.

20.6.4 Show, first, that in any perfect rectangle the smallest constituent square is not on the boundary of the rectangle. Now suppose that there is a perfect cube, and consider the perfect square induced on the base of this cube by the constituent cubes.

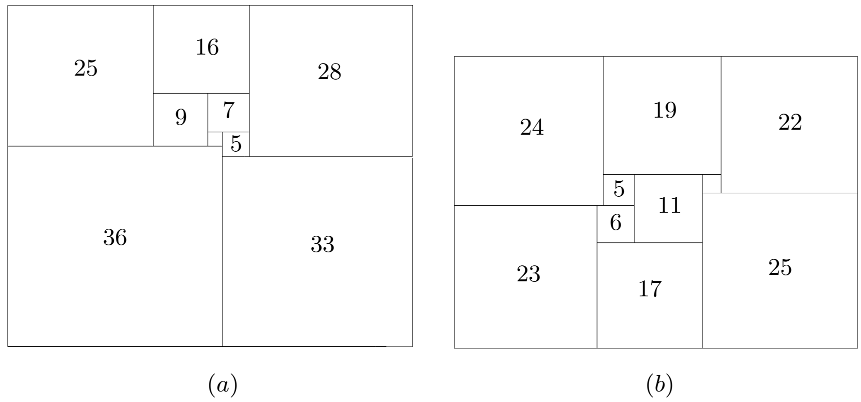

Fig. B.2. Solution to Exercise 20.6.1

20.6.5

(a) A Fibonacci tiling is shown in Figure B.3.

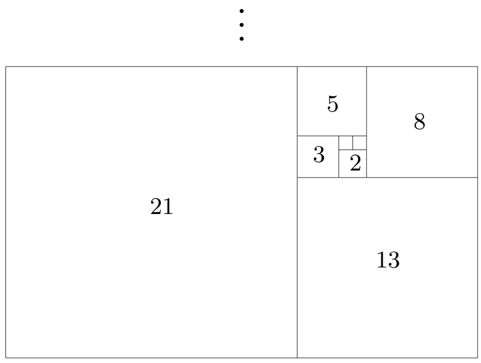

Fig. B.3. Fibonacci tiling of the plane

(b) The Fibonacci tiling is not perfect because there are two squares of size one. To remedy this situation, first obtain, by blowing up the Fibonacci tiling 112 times, a dissection in which there is one square of size $112F_i$, for $i = 1, 2, 3, \ldots$. In this dissection there are two squares of size 112, but the sizes of all others squares are greater than 112 and distinct. Now replace one of the squares of size 112 by the perfect square shown in Figure 20.10.

21.4.7 By Theorem 21.17, there are two edge-disjoint spanning trees, T and T'. Let X be the set of odd vertices of T. By Exercise 3.3.5b, there are $|X|/2$ edge-disjoint paths in T' connecting the elements of X in pairs. The union of T and these paths is a connected even spanning subgraph.

21.2.9 (a) Necessity follows on taking X as the set of vertices with indegree k and Y as the set of vertices with indegree l. To prove sufficiency, construct a network N by forming the associated digraph of G, assigning unit capacity to each arc, and regarding the elements of X as sources and the elements of Y as sinks. By Theorem ???, there is a flow f in N (which can be assumed integral) in which the supply at each source and the demand at each sink is $|k-l|$. The f-saturated arcs induce a (k,l)-orientation on a subgraph H of G. A (k,l)-orientation of G can now be obtained by giving the remaining edges an eulerian orientation.

21.2.13 Let S be any subset of A. By Exercise 21.2.12, the number of circulations in the spanning subdigraph with arc set S is $k^{m-n+c(S)}$. Thus the number of circulations whose supports are included in S is $k^{m-n+c(S)}$. Equivalently, the number of circulations which take value zero on all the arcs of $A(D) \setminus S$ is $k^{m-n+c(S)}$. Since nowhere-zero circulations are those circulations which take the value zero on no arc, we obtain the desired result by applying the Inclusion-Exclusion Formula (2.3).

21.3.2

(a) The proof is straightforward if the degree of v is two. So, suppose that $d(v)=4$ or $d(v)\geq 6$. The required assertion is equivalent to showing that there exists a pair of edges incident with v such that there is no 5-edge cut of G which contains both of them. Towards this end, let $\mathcal{F}$ denote the set of all subsets of $\partial(v)$ such that every member of $\mathcal{F}$ is contained in a 5-edge cut of G. Let F be a maximal member of $\mathcal{F}$ and let X be a subset of $V(G)$, with $v \in X$ and $|\partial(X)| = 5$. We may assume that $F \neq \emptyset$; otherwise, any two edges incident with v would serve the purpose. First show that F is a proper subset of $\partial(v)$. Then let f be an edge in F and let e is an edge in $\partial(v) \setminus F$ and consider any edge cut $\partial(Y)$ of G containing $\{e,f\}$. Show that $d(Y) \neq 5$. (L.M. de Almeida e Silva)

(b) Let G be a graph with no 1- or 3-edge cuts such that (i) G has no 3-flow, and (ii) subject to (i), $v(G)+e(G)$ is as small as possible. Show that G is a 4-edge-connected 5-regular graph.

21.3.3

21.3.4

(a) A 2-edge-connected graph with a non-trivial 2-edge cut has at least two edges. The only such graph with exactly two edges has two vertices with two edges joining them. Verify the statement in this case. To prove the statement in general, apply induction by contracting and deleting edges not belonging to $\partial(X)$. (Note that, in order to apply induction using deletions and contractions, it is necessary to permit loops and multiple edges.)

(b) The proof is similar to that of (a).

21.5.2 First apply Exercise 21.3.1 to show that there exists a 2-edge-connected cubic graph G' with $v(G')+e(G') \leq v(G)+e(G)$, such that G' does not have a k-flow. Then apply Exercise 21.3.4.

21.4.5 Consider an arbitrary orientation D of G. Suppose that there exists a vertex x whose outdegree in D is greater than k. Let X be the set of all vertices reachable from x in D. Show that X includes a vertex y whose outdegree is smaller than k. Now reverse all arcs on some directed (x, y)-path in $D[X]$ and observe that the quantity $\sum_{v \in V} \max\{0, (d^+(v) - k)\}$ is thereby reduced.